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Non-preemptive scheduling algorithms and schedulability conditions for real-time systems with precedence and latency constraints

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Abstract: First we justify our concern in l tency constr ints for re l-time systems with precedence constr ints. We evoke the model b sed on gr ph theory used to st te nd solve the problem of non-preemptive scheduling systems with precedence nd l tency constr ints. Bec use l tency constr ints involve p irs of oper tions rel ted to the gr ph which models the precedence constr ints we define three rel tions between p irs of oper tions rel tions which determine the w y of scheduling the oper tions. We give necess ry nd sufficient condition of schedul bility for the p rticul r c se of systems with precedence nd l tency constr ints where ll p irs on which l tency constr ints re imposed re in one of these rel tions. Fin lly we give gener l schedul bility condition for re l-time systems with precedence nd l tency contr ints nd we propose scheduling lgorithm which is proved optim l.

Key-w rds: re l-time gr phs scheduling schedul bility lgorithms complexity mono-processor

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Ordonnancement non-préemptif et conditions d'ordonnançabilité pour systèmes temps réel avec contraintes de précédences et de latences

Résumé : On commence p r justifier notre intêret pour les contr intes de l tences pour des systèmes temps réel vec contr intes de précédences. On r pelle notre modèle utilis nt les gr phes fin de poser et de résoudre le problème d'ordonn ncement non-préemptif de systèmes vec contr intes de précédences et de l tences. Puisque les contr intes de l tence sont imposées sur des p ires d'opér tions du gr phe qui définit les contr intes de précédence on définit trois rel tions entre p ires d'opér tions rel tions qui détermine l m nière dont les opér tions vont être ordonn ncées. On donne une condition nécess ire et sufis nte d'ordonn nç bilité pour le c s des sytèmes vec contr intes de précédences et de l tences où les p ires vec contr intes de l tences sont en une de trois rel tions. Fin lement on donne une condition génér le d'ordonn nç bilité et on propose un lgorithme d'ordonn ncement qui est prouvé optim l.

M ts-clés : temps réel gr phes ordonn ncement ordonn nç bilité lgorithmes complexité monoprocesseur

1 Introduction

Re l-time systems re first of ll re ctive systems [1] th t is to e ch event rriving in the system through sensors corresponds n output event produced through ctu tors to the environment in re ction to this input event. The rriv l of input events is indefinitly repe ted nd n output event is obt ined from one or sever l input events fter sever l oper tions on which precedence constr ints m y be imposed. For ex mple in the c se of br ke control in c r when the driver presses the br ke ped l the wheels must stop before fixed del y corresponding to the execution of control l ws rel ting "ped l-press" event nd "wheels-stop" event. We c ll this constr int *l tency* constr int. This type of del y w s lre dy used by the re l-time community but either the models took into ccount only one glob l l tency constr int [2 3] or the del y m y be imposed only on sensors nd ctu tors [4]. This p per gener lises these results llowing to h ve sever l l tency constr ints which re not imposed only on sensors nd ctu tors but lso on the oper tions which re necess ry to obt in output events from input events. For ex mple it is possible to specify the br ke system of c r (figure 1) by gr ph where $PRESS$ represents sensor giving the inform tion which indic tes th t the driver presses the br ke ped l $SPEED$ represents sensor giving the wheel speed $DEAC$ represents the de ctiv tion of the speed regul tor $ESTIM$ represents the estim tion of the pressure on the br ke ped l ABS represents the nti-locking br ke system (ABS) nd $BRAKE$ represents the ctu tor giving the pressure on the br kes wheels. We m y need to impose del y between $PRESS$ nd $BRAKE$ nd nother del y between $DEAC$ nd $BRAKE$.

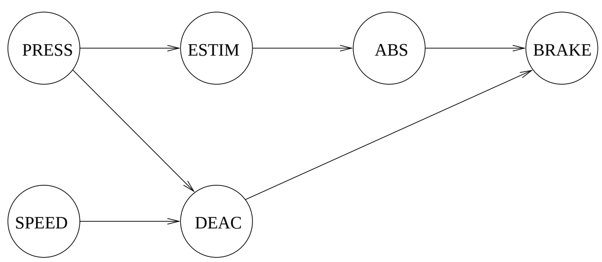


Figure 1 Ex mple of br ke control

In this p per we study the non-preemptive c se nd we pl n to extend subsequently the results to the preemptive c se ev lu ting ccur tely its overhe d.

The next section gives some not tions nd existing results for systems with precedence constr ints. Section 3 presents our model given in [5]. In this section we define the l tency constr int. In the following section we give schedul bility condition for re l-time systems with precedence nd l tency constr ints. Section 5 presents n optim l scheduling lgorithm. The l st section concludes nd proposes future work.

2 Notations and existing results

In order to cle rly distinguish the specific tion level from the implement tion level we use the term *oper tion* inste d of the commonly used “t sk” too closely rel ted to the implemen- t tion level nd in turn we use the term *oper tor* inste d of “processor” or “m chine”.

For n oper tion A we m y specify comput tion time C_{A} de dline D_{A} (defined from the st rt of the schedule) nd st rt time s_{A}. Also for p irs of oper tions we m y specify precedence constr ints th t m y be represented by directed cyclic gr ph where the vertices re the oper tions nd the edges re the precedence constr ints.

We evoke now some m in results on systems with precedence constr ints nd de dlines in the c se of one oper tor. For the non-preemptive problem with precedence constr ints minimising monotone function which llows to define de dlines L wler gives the optim l "first-to-l st-rule" [6]. Also for the non-preemptive problem with precedence nd de dlines Bl zewicz gives polynomi l solution [7]. For the preemptive problem with precedence constr ints nd de dlines there re results on the schedul bility of system [8] [9] [10] nd [11].

In this p per we give schedul bility condition for re l-time systems with precedence nd l tency constr ints using the model proposed in [5].

3 Model for systems with precedence and latency constraints

In our model system with precedence constr ints is defined by infinite directed gr ph $= (\ ,\mathcal{E})$ where is the set of oper tions nd $\mathcal{E}$ the set of edges which represents the precedence constr ints between oper tions. Bec use re l-time system is re ctive we model the system by n infinite gr ph issued from n infinite repetition of finite subgr ph c lled “p ttern” [3]. If one consider only the p ttern itself ccording to its p rti l order the first oper tions re c lled inputs nd the l st oper tions re c lled outputs. These oper tions correspond to respectively sensors nd ctu tors. The edges between oper tions belonging to the s me p ttern re c lled intr -p ttern nd the edges between oper tions belonging to di erent p tterns re c lled extr -p ttern. For ex mple in figure 2 the edge from A to B is n intr -p ttern edge nd the edge from D belonging to the i^{th} p ttern to A belonging to the $(i+1)^{\mathsf{th}}$ p ttern is n extr -p ttern. The infinite repetition of the p ttern of the gr ph induces n infinite repetition of ll the oper tions.

Definiti n 1 *[12] A directed p th from A_1 to A_{n} is n ltern ting sequence $A_1(A_1, A_2)$ $A_2 \ldots (A_{\mathsf{n}-1}, A_{\mathsf{n}})A_{\mathsf{n}}$ of vertices nd distinct edges, where A_{i} nd $(A_{\mathsf{i}}, A_{\mathsf{i}+1})$ $\mathcal{E}$, i $1, 2, \ldots, n$.*

For the s ke of simplicity nd bec use we use only directed gr phs from now on we will use the term p th inste d of directed p th.

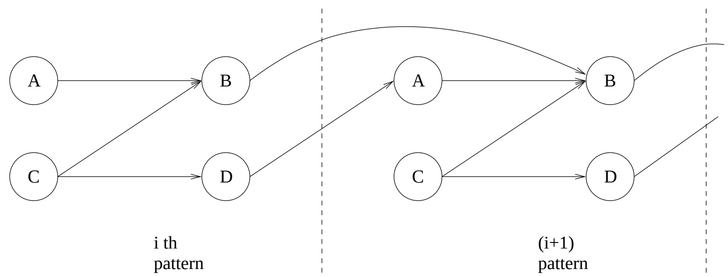


Figure 2 System of oper tions defined by infinite directed gr ph

We denote by $\mathcal{P}$ the set of ll p ths of gr ph nd we s y th t $P(A,B)$ $\mathcal{P}$ if there is t le st p th from A to B. If $\nexists P(A,B)$ $\mathcal{P}$ then there is no p th from A to B. If there is t le st p th from A to B we denote by $\mathcal{M}(A,B)$ the set of ll oper tions belonging to ll the p ths from A to B $\mathcal{M}(A,B) = C$ with $P(A,C)$ $\mathcal{P}$ nd $P(C,B)$ $\mathcal{P}$ $\bigcup$ A, B . In order to illustr te the l tter not tion we give the following ex mple.

Example 1 *Let be the gr ph given by gure 3. The set* $\mathcal{M}(A,G) = A, B, D, C, G$ *is not empty bec use there is t le st p th from* A *to* G *nd we h ve* $P(A,G)$ $\mathcal{P}$. *For ex mple, from* A *to* G *there re four p ths:* $A(A,D)D(D,G)G$, $A(A,B)B(B,D)D(D,G)G$, $A(A,D)D(D,C)C(C,G)G$ *nd* $A(A,B)B(B,D)D(D,C)C(C,G)G$. *On the contr ry the set* $\mathcal{M}(C,E)$ *is empty bec use there is no p th from* C *to* E.

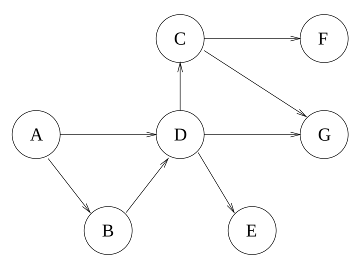


Figure 3 Set of oper tions belonging to ll p ths from A to G

Definiti n 2 *For system of oper tions, schedule* S *is tot l ordered set of st rt times of ll the oper tions* $S = s_{\mathbf{A}}$ $\mathbb{N}, A$ *such th t the precedence constr ints re s tis ed. We denote by the set of ll schedules.*

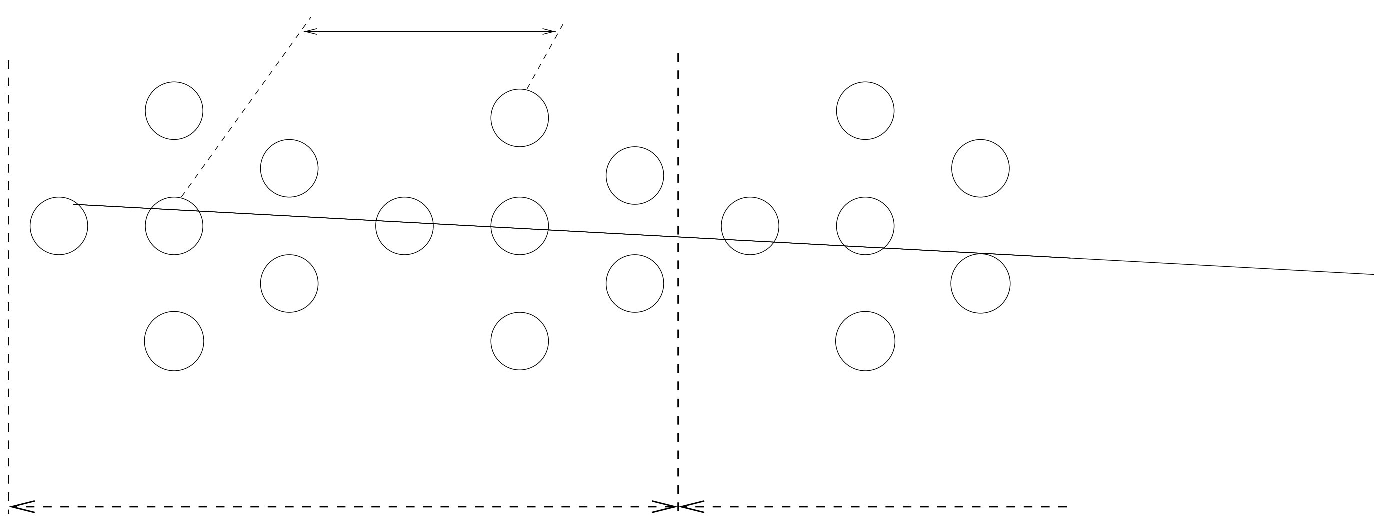
Remark 1 *An oper tion is v il ble when ll its predecessors re scheduled. We consider schedules without idle time, i.e. we schedule v il ble oper tions t the end of the l st scheduled oper tion nd we h ve* A , B *such th t* $s_{\mathbf{A}} + C_{\mathbf{A}} = s_{\mathbf{B}}$.

The edges of system of oper tions define p rti l order on the set of oper tions. There re sever l tot l orders which m y be obt ined from this p rti l order. Bec use schedule is tot l order on the set of oper tions with st rt time ssoci ted to e ch oper tion then we m y h ve sever l schedules for the s me system of oper tions.

Remark 2 *The directed p ir of oper tions* (A, B) $\mathcal{E}$ *implies th t* B *must be scheduled, only if* A *w s lre dy scheduled, nd we h ve* $s_{\mathsf{A}} + C_{\mathsf{A}}$ s_{B}, *where* $s_{\mathsf{A}}, s_{\mathsf{B}}, C_{\mathsf{B}}$ $\mathbb{N}^{+}$.

Definiti n 3 *[5] on p ir of oper tions* (A, B) *with* $P(A, B)$ $\mathcal{P}$ *nd* A *nd* B *belonging to the s me p ttern, we s y th t l tency constr int* $L(A, B)$ *is imposed if the oper tions re scheduled such th t* $s_{\mathsf{B}} + C_{\mathsf{B}}$ s_{A} L_{AB}, L_{AB} $\mathbb{N}^{+}$. *Let* $\mathcal{L}$ *be the set of ll l tency constr ints imposed for system.*

Remark 3 *The condition imposing to* A *nd* B *to belong to the s me p ttern is not restrictive. If we w nt to impose l tency constr int on two oper tions which do not belong to the s me p ttern, then it is sufficient to de ne new p ttern. Indeed bec use the v lue of l tency constr int is nite (otherwise it does not de ne re l-time constr int) we will never obt in n in nite p ttern. The gure 4 gives such n ex mple. We w nt to de ne the l tency constr int* $L(C, B)$ *with* C *nd* B *belonging to two different p tterns. Therefore we choose new p ttern cont ining two old p tterns.*



Definiti n 4 *if there is schedule for system of oper tions which s tis es ll the l tency constr ints then the system is c lled schedul ble.*

The definition 3 gener lises the notion of l tency constr int for two oper tions which re not necess rily input nd output oper tions.

4 Schedulability condition

We give necess ry nd sufficient condition for system with precedence nd l tency constr ints to be schedul ble. Bec use the gr ph is p ttern infinitely repe ted nd the l tency constr ints re imposed on oper tions belonging to the s me p ttern studying only the p ttern is enough to obt in the schedul bility condition.

The l tency constr ints re imposed on p irs of oper tions rel ted by p th which imposes tot l order on the oper tions belonging to this p th. So if we w nt to schedule system of oper tions with n l tency constr ints we study the existence of p ths between p irs of oper tions with l tency constr ints. Consequently in this section first we present rel tions between two p irs of oper tions rel tively to scheduling. For these p irs on which l tency constr ints re imposed we present schedul bility condition.

For the s ke of simplicity we c ll " n oper tion belonging to p ir (X, Y)" n oper tion belonging to t le st one of the p ths from X to Y. Also we s y th t " p ir (X, Y) cont ins n oper tion Z" when Z belongs to the p ir (X, Y).

For some p ir of oper tions (X, Y) belonging to system nd schedule of this system we c ll "schedule length" the time between the st rt of n oper tion X nd the end of n oper tion Y. We denote this time by $\delta_{\mathbf{XY}}(S) = s_{\mathbf{Y}} + C_{\mathbf{Y}} \quad s_{\mathbf{X}}$ where $s_{\mathbf{X}}, s_{\mathbf{Y}} \quad S$. Without ny loss of gener lity we consider th t X is lw ys scheduled before Y.

Bec use we consider th t the v il ble oper tions re scheduled t the end of the l st scheduled oper tion (see rem rk 1) we h ve $\delta_{\mathbf{XY}}(S) = \sum_{\forall \mathbf{C}; \mathbf{s}_X \leq \mathbf{s}_C \leq \mathbf{s}_Y} C_{\mathbf{C}}$.

The rem 1 *Let* $\;= (\;, \mathcal{E})$ *be gr ph de ning system nd* $L(A, B)$ *l tency constr int where* A, B *. If the system is schedul ble then the sum of the execution times of oper tions belonging to* $\mathcal{M}(A, B)$ *is sm ller or equ l to* $L_{\mathbf{A;B}}$*.*

Pr f If the system is schedul ble then there is t le st schedule S s tisfying ll constr ints of the system. So for schedule S we h ve

$$\delta_{\mathbf{AB}}(S) \quad L_{\mathbf{A;B}} \tag{1}$$

Bec use of the p rti l order the oper tions belonging to $\mathcal{M}(A, B)$ must be scheduled between A nd B nd we h ve $\sum_{\mathbf{C} \in \mathcal{M}(\mathbf{A;B})} C_{\mathbf{C}} \quad \delta_{\mathbf{AB}}(S) \quad L_{\mathbf{A;B}}$. So $\sum_{\mathbf{C} \in \mathcal{M}(\mathbf{A;B})} C_{\mathbf{C}}$ $L_{\mathbf{A;B}}$. The theorem is proved ⊡

The theorem 1 gives necess ry condition for l tency constr int to be s tisfied. By imposing this condition we gu r ntee th t we c n lw ys schedule between the first nd the l st oper tion of l tency constr int t le st the oper tions belonging to this l tency constr int.

If between these oper tions there is no oper tion belonging to nother p ir on which l tency constr int is imposed then this condition is lso sufficient (see theorem 2). This is possible bec use ll schedules s tisfying the p rti l order defined by gr ph for which we obt in the sm llest time between the first nd the l st oper tion of the l tency constr int cont in only oper tions belonging to the p ir on which the l tency constr int is imposed. All these schedules le d to the s me set of oper tions scheduled between the first oper tion nd the l st oper tion of the l tency constr int set th t we c ll "minim l set corresponding to the l tency constr int".

On the contr ry if between these oper tions there is t le st one oper tion belonging to nother p ir on which l tency constr int is imposed then the condition given by theorem 1 is only necess ry (see theorem 5). This is possible bec use ll schedules s tisfying the p rti l order defined by gr ph for which we obt in the sm llest time between the first nd the l st oper tion of the l tency constr int m y cont in lso oper tions belonging to nother p ir on which l tency constr int is imposed. All these schedules le d to di erent sets of oper tions scheduled between the first oper tion nd the l st oper tion of the l tency constr int sets th t we c ll "minim l sets corresponding to the l tency constr int".

The first consequence of these interpret tions of theorem 1 is the coroll ry 1 concerning the c se of system with only one l tency constr int. So we re in the c se of l tency constr int imposed on p ir which does not cont in ny oper tion belonging to nother p ir on which l tency constr int is imposed nd the condition given by theorem 1 is necess ry nd sufficient.

C r llary 1 *Let* $= (,\mathcal{E})$ *be gr ph de ning system nd* (A, B) *the only l tency constr int of the system. The system is schedul ble if nd only if the sum of execution times of oper tions belonging to* $\mathcal{M}(A, B)$ *is sm ller or equ l to* $L_{\mathbf{A;B}}$.

Pr f If we h ve only one l tency constr int then studying the schedul bility of the system mounts to consider ll schedules s tisfying the p rti l order nd cont ining between the first nd the l st oper tion of the l tency constr int only oper tions belonging to the p ir on which the l tency constr int is imposed. Indeed for these schedules we obt in the sm llest time between the first nd the l st oper tion of the l tency. Consequently it is sufficient to comp re the sum of execution times of oper tions belonging to the p ir to v lue of the l tency constr int. The coroll ry is proved ⊡

Example 2 *This ex mple illustr tes the c se of one l tency constr int. For the gr ph given in gure 5 we consider the l tency constr int* $L(A, D)$. *We consider ll execution times equ l to* 1. *The minim l set* A, B, C, E, D *cont ins between the rst nd the l st oper tion of the l tency constr int only oper tions belonging to the p ir on which the l tency constr int is imposed. The oper tion* F *does not belong to this set. The system is schedul ble if nd only if the v lue of the l tency constr int is gre ter th n* 5.

4.1 Relations between two pairs of operations relatively to scheduling

We consider two p irs of oper tions nd we study the c se where they re connected or not by p ths. The p ths which connect two p irs re those who st rt from n oper tion belonging to one of the p irs to n oper tion belonging to the other p ir.

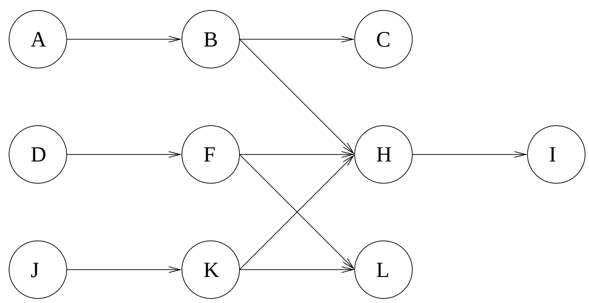


Figure 6 Ex mple of rel tions between p irs of oper tions

Thus for two p irs of oper tions we h ve 2^2 possible combin tions covered by the three following c ses

1. there is no p th from n oper tion belonging to one of the p ths between the considered p ir to n oper tion belonging to one of the p ths between the other p ir (for ex mple p irs (A, C) nd (J, L) of figure 6). This rel tion is c lled II;
2. there is t le st p th from one p ir to the other one nd there is no p th from the l tter p ir to the first one or vice vers (for ex mple p irs (A, C) nd (D, I) of figure 6). This rel tion is c lled Z. Note th t it covers two c ses;
3. there is t le st p th from one p ir to the other one nd vice vers (for ex mple p irs (D, I) nd (J, L) of figure 6). This rel tion is c lled X.

In the c se of rel tion II (see figure 6) no precedence constr int imposes to n oper tion belonging to $\mathcal{M}(A, C)$ to be scheduled before or fter ll oper tions belonging to $\mathcal{M}(J, L)$. For ex mple the schedules with A B nd C scheduled before J K nd L or vice vers s tisfy the p rti l order.

In the c se of rel tion Z (see figure 6) the precedence constr ints impose to oper tions belonging to $\mathcal{M}(A, B)$ to be scheduled before oper tion belonging to $\mathcal{M}(H, I)$. However they do not impose ny order between oper tions belonging to $\mathcal{M}(B, C)$ B nd oper - tions belonging to $\mathcal{M}(D, H)$ H . For ex mple A B nd C m y be scheduled before ll oper tions belonging to $\mathcal{M}(D, I)$ nd simil rly D F H nd I m y be scheduled before ll oper tions belonging to $\mathcal{M}(A, C)$.

In the c se of rel tion X (see figure 6) the precedence constr ints impose to oper tions belonging to $\mathcal{M}(D,F)$ to be scheduled before oper tions belonging to $\mathcal{M}(L,L)$ nd to oper tions belonging to $\mathcal{M}(H,I)$ to be scheduled before oper tions belonging to $\mathcal{M}(J,K)$. However they do not impose ny order between oper tions belonging to $\mathcal{M}(D,F)$ nd oper tions belonging to $\mathcal{M}(J,K)$. For ex mple D nd F m y be scheduled before or fter J nd K nd H nd I must be scheduled fter J nd K nd vice vers .

By t king into ccount the rel tions between p irs of oper tions we study the schedul bility condition for l tency constr ints imposed on p irs in rel tion Z nd/or in rel tion $\parallel$. We study the schedul bility condition for l tency constr ints imposed on p irs in rel tion X. Fin lly we give gener l schedul bility condition for l tency constr ints imposed on p irs in rel tion $\parallel$ Z nd X.

4.2 Relations $\parallel$ and Z

We study the schedul bility of systems with l tency constr ints imposed on p irs in rel tion Z nd/or in rel tion $\parallel$. For ex mple in figure 7 we h ve three l tency constr ints $L(A,C,)$ $L(D,I)$ nd $L(J,L)$ imposed on p irs in rel tion Z nd/or in rel tion $\parallel$.

Only the rel tion Z imposes p rti l order on p irs where s the rel tion $\parallel$ does not impose ny order. For ex mple the p rti l order imposes only to A nd to B to be scheduled before H nd I.

The schedul bility studies se rch mong ll possible schedules those le ding to minim l sets corresponding to l tencies. The lemm 1 llows to find those schedules nd lemm 2 proves th t for p irs in rel tion $\parallel$ nd in rel tion Z those minim l sets re identic l. Consequently even if rel tively to the p rti l order the two c ses re not identic l rel tively to the schedul bility they re. This implies th t we m y give n unique schedul bility condition for the two rel tions. This condition is given by theorem 2 which ends this subsection.

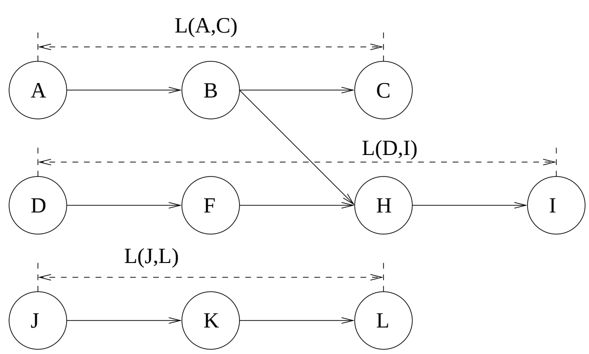


Figure 7 L tency constr ints

Lemma 1 *Let A, B be two oper tions belonging to system given by gr ph $= (\ , \mathcal{E})$, such th t $P(A, B) \quad \mathcal{P}$. We denote by S schedule such th t $H \quad \mathcal{M}(A, B)$, either we h ve $s_{\mathbf{H}} + C_{\mathbf{H}} < s_{\mathbf{A}} + C_{\mathbf{A}}$, or we h ve $s_{\mathbf{B}} + C_{\mathbf{B}} < s_{\mathbf{H}} + C_{\mathbf{H}}$. If $\ '$ is the set of the schedules S' such th t $H \quad \mathcal{M}(A, B)$ with $s'_{\mathbf{A}} + C_{\mathbf{A}} < s'_{\mathbf{H}} + C_{\mathbf{H}} < s'_{\mathbf{B}} + C_{\mathbf{B}}$, then $\delta_{\mathbf{AB}}(S) \quad \delta_{\mathbf{AB}}(S')$, $S' \quad '$.*

Pr f Bec use S is schedule such th t $H \quad \mathcal{M}(A, B)$ either we h ve $s_{\mathbf{H}} + C_{\mathbf{H}} < s_{\mathbf{A}} + C_{\mathbf{A}}$ or we h ve $s_{\mathbf{B}} + C_{\mathbf{B}} < s_{\mathbf{H}} + C_{\mathbf{H}}$ only oper tions belonging to $\mathcal{M}(A, B)$ re scheduled from $s_{\mathbf{A}}$ to $s_{\mathbf{B}}$ nd we h ve

$$\delta_{\mathbf{AB}}(S) = \sum_{\mathbf{I} \in \mathcal{M}(\mathbf{A};\mathbf{B})} C_{\mathbf{I}} \tag{2}$$

For some schedule $S' \quad '$ let $N = \quad H$ such th t $H \quad \mathcal{M}(A, B)$ nd $s'_{\mathbf{A}} + C_{\mathbf{A}} < s'_{\mathbf{H}} + C_{\mathbf{H}} < s'_{\mathbf{B}} + C_{\mathbf{B}}$. We h ve

$$\delta_{\mathbf{AB}}(S') = \sum_{\mathbf{I} \in \mathcal{M}(\mathbf{A};\mathbf{B})} C_{\mathbf{I}} + \sum_{\mathbf{H} \in \mathbf{N}} C_{\mathbf{H}}, \quad S' \quad ' \tag{3}$$

By subtr cting the equ tions (2) nd (3) we obt in $\delta_{\mathbf{AB}}(S') \quad \delta_{\mathbf{AB}}(S) = \sum_{\mathbf{H} \in \mathbf{N}} C_{\mathbf{H}}$ 0, S' . Fin lly we h ve $\delta_{\mathbf{AB}}(S) \quad \delta_{\mathbf{AB}}(S') \quad S' \quad '$. The lemm is proved $\boxdot$

Lemma 2 *For four oper tions A, B, C nd D , if $(A, B) \parallel (C, D)$ or if $(A, B)\ \mathsf{Z}\ (C, D)$ then we h ve $\mathcal{M}(A, B) \bigcap \mathcal{M}(C, D) = \ $.*

Pr f We prove by contr diction th t if $(A, B)\ \mathsf{Z}\ (C, D)$ then $\mathcal{M}(A, B) \bigcap \mathcal{M}(C, D) = \ $.

We suppose th t $F \quad \mathcal{M}(A, B) \bigcap \mathcal{M}(C, D)$. Bec use $F \quad \mathcal{M}(A, B)$ respectively F $\mathcal{M}(C, D)$ we h ve

$$P(A, F) \quad \mathcal{P} \tag{4}$$

nd

$$P(F, B) \quad \mathcal{P} \tag{5}$$

respectively

$$P(C, F) \quad \mathcal{P} \tag{6}$$

nd

$$P(F, D) \quad \mathcal{P} \tag{7}$$

From (4) nd (7) respectively from (5) nd (6) we obt in $P(A, D) \quad \mathcal{P}$ nd $P(C, B)$ $\mathcal{P}$. This is in contr diction with the definition of rel tion Z which s ys th t there is t le st one p th from p ir to the other p ir nd there is no p th from the l st p ir to the first p ir. We h ve contr diction so the supposition th t there is $F \quad \mathcal{M}(A, B) \bigcap \mathcal{M}(C, D)$ is not true nd we h ve $\mathcal{M}(A, B) \bigcap \mathcal{M}(C, D) = \ $.

The proof for the c se of rel tion $\parallel$ is simil r to c se of rel tion X. The lemm is proved $\boxdot$

The lemm 2 proves th t l tency constr int $L(A,B)$ in rel tion II or in rel tion Z with nother l tency constr int $L(C,D)$ does not cont in ny oper tion belonging to this l tter constr int ($\mathcal{M}(A,B) \bigcap \mathcal{M}(C,D) =$). This implies th t e ch minim l set corresponding to l tency constr int cont ins only oper tions belonging to this constr int so for the two c ses the minim l set corresponding to e ch l tency constr int is identic l.

The rem 2 *Let = (,$\mathcal{E}$) be gr ph de ning system nd e ch two p irs, on which l tency constr ints re imposed, re either in rel tion* II*, either in* Z*. A schedule s tis es ll the l tency constr ints if nd only if* $\sum_{H \in \mathcal{M}(A;B)} C_H \quad L_{AB}$*, for ll A nd B h ving l tency constr int.*

Pr f The lemm 1 shows th t for two oper tions A nd B rel ted by p th scheduling only oper tions which belong to $\mathcal{M}(A,B)$ between A to B will decre se the v lue of $\delta_{AB}(.)$. This me ns th t if schedule in which only oper tions which belong to $\mathcal{M}(A,B)$ re scheduled between A to B do not s tisfy l tency constr int L_{AB} then no other schedule will. Therefore nd bec use $\mathcal{M}(A,B) \bigcap \mathcal{M}(C,D) =$ for ll p irs which re in rel tion II or in rel tion Z (lemm 1 nd lemm 2) from now on we will consider only schedules of this type.

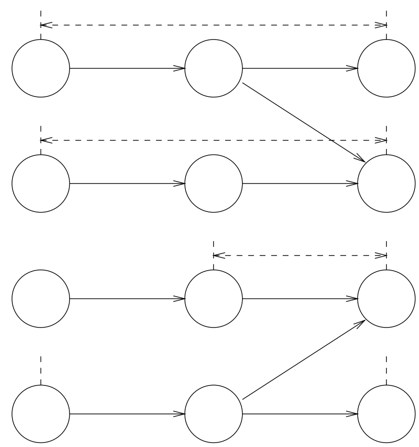
A schedule S s tisfies ll the l tency constr ints if nd only if $\delta_{AB} \quad L_{AB} \quad A$ nd B h ving l tency constr int (definition 3). Bec use $H \quad \mathcal{M}(A,B)$ either we h ve $s_H + C_H < s_A + C_A$ or we h ve $s_B + C_B < s_H + C_H$ then schedule S s tisfies ll the l tency constr ints if nd only if $\sum_{H \in \mathcal{M}(A;B)} C_H \quad L_{AB}$ for ll A nd B h ving l tency constr int. The theorem is proved $\boxdot$

Remark 4 *When the l tency constr ints re in rel tion* II *or in rel tion* Z*, the minim l set corresponding to l tency constr int is equ l to the set of oper tions belonging to this l tency constr int. The rst p rt of ex mple 3 illustr tes this kind of set.*

Example 4 *This ex mple illustr tes the schedul bility condition for system of oper tions with l tency constr ints in rel tion* II *or in rel tion* Z*. For the gr ph given in gure we consider the l tency constr ints $L(A,C)$, $L(D,F)$, $L(H,I)$ nd, respectively, $L(J,L)$ imposed on p irs (A,C), (D,F), (H,I) nd, respectively, (J,L). All l tency constr ints re equ l to 3 nd $C_A = 1$, A . We notice th t ll p irs re either in rel tion* II*, either in rel tion* Z*. The schedul bility condition implies th t l tencies must be gre ter th n the sum of execution times of oper tions. We suppose th t this condition is s tis ed so there is t le st one schedule which s tis es every constr int of the system. Figure 9 gives such schedule.*

4.3 Relation X

In order to present the results concerning systems with l tency constr ints imposed on p irs in rel tion X we introduce two new not tions.



Let $(A,B),(C,D)$ with $P(A,B),P(C,D)$ $\mathcal{P}$ be some p irs in rel tion X. E ch p ir h s oper tions which must be scheduled before oper tions belonging to the other p ir. Like in the previous c se of p irs in rel tion II nd/or in rel tion Z the schedul bility studies for p irs in rel tion X re b sed on minim l sets corresponding to the l tency constr ints.

For two p irs in rel tion X ft denotes set of oper tions which re first on ll p ths between the first oper tion of p ir nd those oper tions comp r tively to the existence of t le st p th from one p ir to these oper tions.

We denote by $ft_{\mathsf{CD}}(A,B)$ the set of oper tions E $\mathcal{M}(A,B)$ for which F $\mathcal{M}(C,D)$ such th t $P(F,E)$ $\mathcal{P}$ nd $\nexists G$ $\mathcal{M}(A,E)$ for which F $\mathcal{M}(C,D)$ such th t $P(F,G)$ $\mathcal{P}$. This me ns th t $ft_{\mathsf{CD}}(A,B)$ is the set of oper tions E belonging to one of the p ths from A to B such th t there is p th to E from one of the oper tions belonging to one of the p ths from C to D nd there is no oper tion G belonging to one of the p ths from A to E such th t there is p th to G from one of the oper tions belonging to one of the p ths from C to D. We denote by $\Gamma^{-}_{\mathsf{CD}}(A,B)$ the set of predecessors of $ft_{\mathsf{CD}}(A,B)$ i.e. the set of oper tions E for which there is n oper tion F $ft_{\mathsf{CD}}(A,B)$ such th t (E,F) $\mathcal{E}$.

For two p irs in rel tion X lt denotes set of oper tions which re l st on ll p ths between those oper tions nd the l st oper tion of p ir nd comp r tively to the existence of t le st p th from these oper tions to the p ir.

We denote by $lt_{\mathsf{CD}}(A,B)$ the set of oper tions E $\mathcal{M}(A,B)$ for which F $\mathcal{M}(C,D)$ such th t $P(E,F)$ $\mathcal{P}$ nd $\nexists G$ $\mathcal{M}(E,B)$ for which F $\mathcal{M}(C,D)$ such th t $P(G,F)$ $\mathcal{P}$. This me ns th t $lt_{\mathsf{CD}}(A,B)$ is the set of oper tions E belonging to one of the p ths from A to B such th t there is p th from E to one of the oper tions belonging to one of the p ths from C to D nd there is no oper tion G belonging to one of the p ths from E to B such th t there is p th from G to one of the oper tions belonging to one of the p ths from C to D. We denote by $\Gamma^{+}_{\mathsf{CD}}(A,B)$ the set of successors of $lt_{\mathsf{CD}}(A,B)$ i.e. the set of oper tions E for which there is n oper tion F $lt_{\mathsf{CD}}(A,B)$ such th t (F,E) $\mathcal{E}$.

Example 5 *In order to illustr te these not tions we give the following ex mple. In gure 10, for (A,B) X (C,D), we h ve $ft_{\mathsf{CD}}(A,B) =$ F,G , $lt_{\mathsf{CD}}(A,B) =$ G , $ft_{\mathsf{AB}}(C,D) =$ G nd $lt_{\mathsf{AB}}(C,D) =$ G,I .*

The schedul bility studies is b sed g in on minim l set belonging to l tency constr int L_1 which must be scheduled before the l st oper tion of nother l tency constr int L_2 with L_1 X L_2. We study in schedule the position of oper tions belonging to L_1 which only incre se the schedule length between the first oper tion nd the l st oper tion of L_2 without modifying the schedule length between the first oper tion nd the l st oper tion of L_1 once the first oper tions or the l st oper tions were scheduled.

For ex mple in figure 11 we consider th t L_1 is l tency constr int on p ir (A,B) nd th t L_2 is l tency constr int on p ir (C,D). If we schedule A before C then for ll positions of oper tion E in the schedule the schedule length between A nd B st ys identic l. On the contr ry the position of E is import nt for the other p ir (C,D). If we schedule E fter C we only incre se the schedule length between C nd D.

Simil rly in figure 12 we consider th t L_1 is the l tency constr int on p ir (A,B) nd th t L_2 is l tency constr int on p ir (C,D). If we schedule B before D then the position of

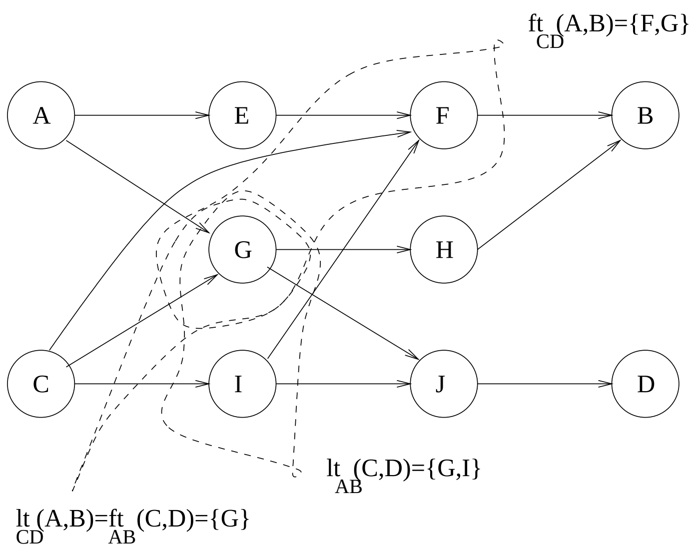


Figure 10 Ex mple for ft nd lt

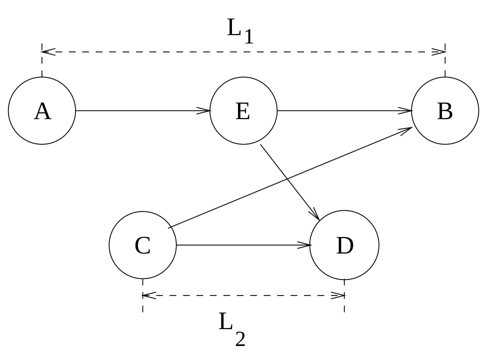


Figure 11 Oper tions only incre sing the schedule length which re t the beginning of l tency constr int

oper tion F in the schedule is not import nt for the p ir (C, D) bec use the schedule length time between C nd D st ys identic l. On the contr ry the position of F is import nt for the other p ir (A, B) bec use if we schedule F before B we incre se the schedule length between A nd B.

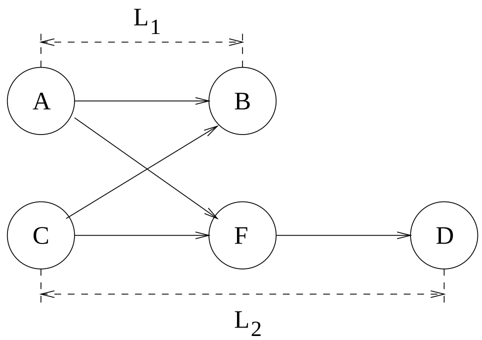


Figure 12 Oper tions only incre sing the schedule length which re t the end of l tency constr int

The rem 3 *Let* $= (\; ,\mathcal{E})$ *be gr ph de ning system nd there re two l tency constr ints imposed on p irs* (A, B) *nd* (C, D)*, where* $(A, B) \; \mathsf{X} \; (C, D)$ *nd* $L_{\mathbf{AB}} = \sum_{\mathbf{H} \in \mathcal{M}(\mathcal{A};\mathcal{B})} C_{\mathbf{H}}$. *The system is schedul ble if nd only if* $L_{\mathbf{CD}} \quad \sum_{\mathbf{H} \in \mathcal{M}(\mathbf{A};\mathbf{B}) \bigcup \mathcal{M}(\mathbf{C};\mathbf{D})} C_{\mathbf{H}}$.

Pr f Bec use $L_{\mathbf{AB}} = \sum_{\mathbf{H} \in \mathcal{M}(\mathcal{A};\mathcal{B})} C_{\mathbf{H}}$ then we m y schedule between the oper tion A nd the oper tion B only oper tions belonging to $\mathcal{M}(A, B)$. So no oper tion belonging to $\mathcal{M}(C, D)$ nd not belonging to $\mathcal{M}(A, B)$ c n be scheduled between A nd B.

Consequently when C nd D do not belong to $\mathcal{M}(A, B)$ these oper tions must be scheduled before A nd fter B which implies th t ll oper tions belonging to $\mathcal{M}(A, B)$ re scheduled between C nd D. And then we h ve $L_{\mathbf{CD}} \quad \sum_{\mathbf{H} \in \mathcal{M}(\mathbf{A};\mathbf{B}) \bigcup \mathcal{M}(\mathbf{C};\mathbf{D})} C_{\mathbf{H}}$. The theorem is proved ⊡

Remark 5 *In theorem 3 we do not t ke into ccount the trivi l c se where* C *nd* D *belong to* $\mathcal{M}(A, B)$*. Indeed if* C *nd* D *belong to* $\mathcal{M}(A, B)$*, then ll oper tions belonging to* $\mathcal{M}(C, D)$ *belong lso to* $\mathcal{M}(A, B)$*. So, the l tency constr int imposed on* (A, B) *must be t le st gre ter or equ l to the sum of execution times of oper tions belonging to* $\mathcal{M}(A, B)$*.*

Remark 6 *For the c se studied by theorem 3 the minim l set of the l tency constr int* (A, B) *is equ l to the set of oper tions belonging to the l tency constr int. For the s me c se, the minim l set of the l tency constr int* (C, D) *is equ l to the set of oper tions belonging to the two l tency constr ints. For ex mple in gure 6 if we consider the l tency constr int* $L(D, I)$ *is equ l to the sum of execution times of oper tions belonging to* $\mathcal{M}(D, I)$ *then the* *minim l set of the constr int* (D, I) *is equ l to* D, F, H, I *nd the minim l set of the constr int* (J, L) *is equ l to* J, K, L, D, F, H, I *.*

The two properties concerning the position of the first nd the l st oper tions of two constr ints re proved by lemm 3 nd lemm 4. The theorem 5 which follows the two lemm uses it for the c se where ll l tency constr ints re in rel tion X.

Lemma 3 *We h ve system given by gr ph* $= (, \mathcal{E})$ *such th t ll p irs, on which l tency constr ints re imposed, re in rel tion* X*. For* (A, B) X (C, D) *nd some schedule* S *with* $s_{\mathbf{A}}$ $s_{\mathbf{C}}$*, if ll oper tions* E $\bigcup_{\mathbf{H}\in\Gamma^-_{CD}(\mathbf{A;B})} \mathcal{M}(A, H)$ *re scheduled between* $s_{\mathbf{A}}$ *nd* $s_{\mathbf{C}}$*, then* $\delta_{\mathbf{AB}}(S)$ *is not modi ed comp red with the c se when t le st n oper - tion* E $\bigcup_{\mathbf{H}\in\Gamma^-_{CD}(\mathbf{A;B})} \mathcal{M}(A, H)$ *is scheduled fter* $s_{\mathbf{C}}$*. Also, if t le st n oper tion* E $\bigcup_{\mathbf{H}\in\Gamma^-_{CD}(\mathbf{A;B})} \mathcal{M}(A, H)$ *is scheduled fter* $s_{\mathbf{C}}$*, then* $\delta_{\mathbf{CD}}(S)$ *is incre sed comp red with the c se when ll oper tions* E $\bigcup_{\mathbf{H}\in\Gamma^-_{CD}(\mathbf{A;B})} \mathcal{M}(A, H)$ *re scheduled between* $s_{\mathbf{A}}$ *nd* $s_{\mathbf{C}}$*.*

Pr f Bec use E $\bigcup_{\mathbf{H}\in\Gamma^-_{CD}(\mathbf{A;B})} \mathcal{M}(A, H)$ $\mathcal{M}(A, B)$ E is lw ys scheduled between $s_{\mathbf{A}}$ nd $s_{\mathbf{B}}$ so $\delta_{\mathbf{AB}}(S)$ will lw ys cont in the comput tion time of E nd the position of E in the schedule does not modify $\delta_{\mathbf{AB}}(S)$.

Bec use E $\bigcup_{\mathbf{H}\in\Gamma^-_{CD}(\mathbf{A;B})} \mathcal{M}(A, H)$ there is no oper tion F $\mathcal{M}(C, D)$ which must be scheduled before E. So E m y be scheduled before C. If we schedule ny oper tion E $\bigcup_{\mathbf{H}\in\Gamma^-_{CD}(\mathbf{A;B})} \mathcal{M}(A, H)$ fter C it is obvious th t for some schedule S $\delta_{\mathbf{CD}}(S)$ is incre sed with the sum of comput tion times of these oper tions E. The lemm is proved $\boxdot$

Lemma 4 *We h ve system given by gr ph* $= (, \mathcal{E})$ *such th t ll p irs, on which l tency constr ints re imposed, re in rel tion* X*. For* (A, B) X (C, D) *nd some schedule* S *with* $s_{\mathbf{D}}$ $s_{\mathbf{B}}$*, if ll oper tions* E $\bigcup_{\mathbf{H}\in\Gamma^+_{CD}(\mathbf{A;B})} \mathcal{M}(H, B)$ *re scheduled between* $s_{\mathbf{D}}$ *nd* $s_{\mathbf{B}}$*, then* $\delta_{\mathbf{AB}}(S)$ *is not modi ed comp red with the c se when t le st n oper - tion* E $\bigcup_{\mathbf{H}\in\Gamma^+_{CD}(\mathbf{A;B})} \mathcal{M}(H, B)$ *is scheduled before* $s_{\mathbf{D}}$*. Also, if t le st n oper tion* E $\bigcup_{\mathbf{H}\in\Gamma^+_{CD}(\mathbf{A;B})} \mathcal{M}(H, B)$ *is scheduled before* $s_{\mathbf{D}}$*, then* $\delta_{\mathbf{CD}}(S)$ *is incre sed comp red with the c se when ll oper tions* E $\bigcup_{\mathbf{H}\in\Gamma^+_{CD}(\mathbf{A;B})} \mathcal{M}(H, B)$ *re scheduled between* $s_{\mathbf{D}}$ *nd* $s_{\mathbf{B}}$*.*

Pr f Bec use $E \quad \bigcup_{\mathbf{H}\in\Gamma^{+}_{CD}(\mathbf{A};\mathbf{B})} \quad \mathcal{M}(H,B) \quad \mathcal{M}(A,B)$ E is lw ys scheduled between $s_{\mathbf{A}}$ nd $s_{\mathbf{B}}$ so $\delta_{\mathbf{AB}}(S)$ will lw ys cont in the comput tion time of E nd the position of E in the schedule does not modify $\delta_{\mathbf{AB}}(S)$. Bec use $E \quad \bigcup_{\mathbf{H}\in\Gamma^{+}_{CD}(\mathbf{A};\mathbf{B})} \quad \mathcal{M}(H,B)$ there is no oper tion $F \quad \mathcal{M}(C,D)$ which must be scheduled fter E. So D m y scheduled before ll oper tions $E \quad \bigcup_{\mathbf{H}\in\Gamma^{+}_{CD}(\mathbf{A};\mathbf{B})} \quad \mathcal{M}(H,B)$. If we schedule ny oper tion $E \quad \bigcup_{\mathbf{H}\in\Gamma^{+}_{CD}(\mathbf{A};\mathbf{B})} \quad \mathcal{M}(H,B)$ before D it is obvious th t for some schedule $S \quad \delta_{\mathbf{CD}}(S)$ is incre sed with the sum of comput tion times of these oper tions E. The lemm is proved $\boxdot$

The rem 4 *Let $\quad = (\quad ,\mathcal{E})$ be gr ph de ning gr ph nd there re two l tency constr ints imposed on the p irs (A,B) nd (C,D), where $(A,B) \mathsf{X} (C,D)$. The two l tency constr ints re not equ l to the sum of execution times of oper tions belonging to the constr int. The system is schedul ble if nd only if*

$$\left\{ \begin{array}{ll} L_{\mathbf{AB}} & \sum_{\mathbf{H}\in\mathcal{M}(\mathbf{A};\mathbf{B})} C_{\mathbf{H}} + \sum_{\mathbf{H}\in\bigcup_{E\in ft_{AB}(C,D)}\mathcal{M}(\mathbf{E};\mathbf{D})} C_{\mathbf{H}} \\ L_{\mathbf{CD}} & \sum_{\mathbf{H}\in\mathcal{M}(\mathbf{C};\mathbf{D})} C_{\mathbf{H}} + \sum_{\mathbf{H}\in\bigcup_{E\in lt_{CD}(A,B)}\mathcal{M}(\mathbf{A};\mathbf{E})} C_{\mathbf{H}} \end{array} \right. \tag{8}$$

or

$$\left\{ \begin{array}{ll} L_{\mathbf{CD}} & \sum_{\mathbf{H}\in\mathcal{M}(\mathbf{C};\mathbf{D})} C_{\mathbf{H}} + \sum_{\mathbf{H}\in\bigcup_{E\in ft_{CD}(A,B)}\mathcal{M}(\mathbf{E};\mathbf{B})} C_{\mathbf{H}} \\ L_{\mathbf{AB}} & \sum_{\mathbf{H}\in\mathcal{M}(\mathbf{A};\mathbf{B})} C_{\mathbf{H}} + \sum_{\mathbf{H}\in\bigcup_{E\in lt_{AB}(C,D)}\mathcal{M}(\mathbf{C};\mathbf{E})} C_{\mathbf{H}} \end{array} \right. \tag{9}$$

Pr f The proof is m de by double implic tion.

First we prove th t if the system is schedul ble then 8 or 9 re s tisfied. Bec use the system is schedul ble then there is t le st one schedule S s tisfying ll constr ints. For this schedule A is scheduled before or fter C.

Bec use we re in the c se of l tency constr ints which re not equ l to the sum of execution times of oper tions belonging to the constr int when A is scheduled before C B is scheduled before D (see figure 13).

Using lemm 3 bec use A is scheduled before C then ll oper tions $E \quad \bigcup_{\mathbf{H}\in\Gamma^{-}_{CD}(\mathbf{A};\mathbf{B})} \quad \mathcal{M}(A,H)$ re scheduled between $s_{\mathbf{A}}$ nd $s_{\mathbf{C}}$. We obt in

$$L_{\mathbf{AB}} \quad \sum_{\mathbf{H}\in\mathcal{M}(\mathbf{A};\mathbf{B})} C_{\mathbf{H}} + \sum_{\mathbf{H}\in\bigcup_{E\in ft_{AB}(C,D)}\mathcal{M}(\mathbf{E};\mathbf{D})} C_{\mathbf{H}} \tag{10}$$

Using lemm 4 bec use B is scheduled before D then ll oper tions $E \quad \bigcup_{\mathbf{H}\in\Gamma^{+}_{AB}(\mathbf{C};\mathbf{D})} \mathcal{M}(H,D)$ re scheduled before $s_{\mathbf{D}}$. We obt in

$$L_{\mathbf{CD}} \quad \sum_{\mathbf{H}\in\mathcal{M}(\mathbf{C};\mathbf{D})} C_{\mathbf{H}} + \sum_{\mathbf{H}\in\bigcup_{E\in lt_{CD}(A,B)}\mathcal{M}(\mathbf{A};\mathbf{E})} C_{\mathbf{H}} \tag{11}$$

From 10 nd 11 we obt in th t 8 is s tisfied.

Bec use we re in the c se of l tency constr ints which re not equ l to the sum of execution times of oper tions belonging to the constr int when A is scheduled fter C B is scheduled fter D (see figure 14) nd the proof follows the s me principles s for the first c se in order to prove th t 9 is s tisfied. The direct implic tion is proved.

We prove now th t if 8 or 9 is s tisfied then the system is schedul ble. The ide of this proof is to build schedule which s tisfies the l tency constr ints. First we suppose th t 8 is s tisfied. We consider the schedule given in figure 13 with ll oper tions $E \quad \bigcup_{\mathbf{H}\in\Gamma^{+}_{AB}(\mathbf{C};\mathbf{D})} \mathcal{M}(H,D)$ scheduled before $s_{\mathbf{D}}$ nd ll oper tions $E \quad \bigcup_{\mathbf{H}\in\Gamma^{-}_{CD}(\mathbf{A};\mathbf{B})} \mathcal{M}(A,H)$ scheduled between $s_{\mathbf{A}}$ nd $s_{\mathbf{C}}$. Bec use of 8 this schedule s tisfies the l tency constr ints.

We suppose th t 9 is s tisfied. The proof follows the s me principles s before using the schedule given in figure 14 in order to prove th t the system is schedul ble. The theorem is proved $\boxdot$

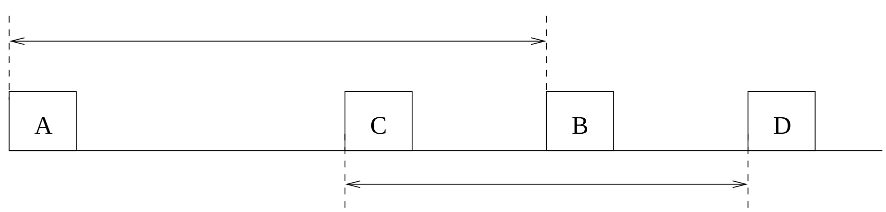


Figure 13 Schedule illustr ting theorem 4

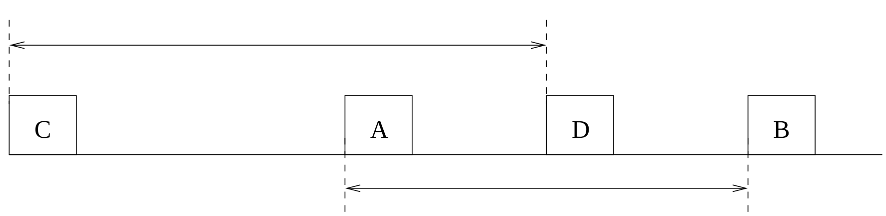


Figure 14 Schedule illustr ting theorem 4

Remark 7 *for the c se studied by theorem 4 the minim l sets corresponding to the two l tency constr ints do not cont in only oper tions belonging to the l tency constr int but lso oper tions belonging to the other l tency constr int. The ex mple 3 illustr tes such sets.*

The rem 5 *Let $= (,\mathcal{E})$ be gr ph de ning system nd ll p irs, on which l tency constr ints re imposed, re in rel tion X. The system is schedul ble if nd only if, for e ch p ir (A, B) on which l tency constr int is imposed, nd which is in rel tion X with n p irs $(C_{\mathbf{i}}, D_{\mathbf{i}})$, i $1, 2, \ldots, n$, on which l tency constr ints re imposed, we h ve*

$$L_{\mathbf{AB}} \quad \sum_{\mathbf{H}\in\mathcal{W}} C_{\mathbf{H}} \tag{12}$$

where:

$$= \mathcal{M}(A,B) \bigcup \bigcup_{\mathbf{i}\in\{1;:::;\mathbf{j}\};\mathbf{E}\in\mathbf{lt}_{AB}(\mathbf{C}_i;\mathbf{D}_i)} \mathcal{M}(C_{\mathbf{i}}, E) \bigcup \bigcup_{\mathbf{i}\in\{\mathbf{j}+1;:::;\mathbf{k}\};\mathbf{E}\in\mathbf{f}\,\mathbf{t}_{AB}(\mathbf{C}_i;\mathbf{D}_i)} \mathcal{M}(E, D_{\mathbf{i}}) \bigcup \bigcup_{\mathbf{i}\in\{\mathbf{k}+1;:::;\mathbf{m}\}} \mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}}),$$

where $j < k < m < n$ re the n tur l numbers for which the following rel tions re s tis ed:

$$L_{\mathbf{C}_i\mathbf{D}_i} \quad \sum_{\mathbf{H}\in\{\mathcal{M}(\mathbf{C}_i;\mathbf{D}_i)\bigcup \bigcup_{\mathbf{E}\in\mathbf{f}\,\mathbf{t}_{C_iD_i}(\mathbf{AB})} \mathcal{M}(E,B)} C_{\mathbf{H}}, \quad i \quad 1, \ldots, j$$

$$L_{\mathbf{C}_i\mathbf{D}_i} \quad \sum_{\mathbf{H}\in\{\mathcal{M}(\mathbf{C}_i;\mathbf{D}_i)\bigcup \bigcup_{\mathbf{E}\in\mathbf{lt}_{C_iD_i}(\mathbf{A};\mathbf{B})} \mathcal{M}(A,E)} C_{\mathbf{H}}, \quad i \quad j+1, \ldots, k$$

$L_{\mathbf{C}_i\mathbf{D}_i} \quad \sum_{\mathbf{H}\in\mathcal{M}(\mathbf{C}_i;\mathbf{D}_i)} C_{\mathbf{H}}, \quad i \quad k+1, \ldots, m$ *nd* $L_{\mathbf{C}_i\mathbf{D}_i} \quad \sum_{\mathbf{H}\in\{\mathcal{M}(\mathbf{C}_i;\mathbf{D}_i)\bigcup\mathcal{M}(\mathbf{A};\mathbf{B})\}} C_{\mathbf{H}}, \quad i$ $m+1, \ldots, n$.

Pr f The proof is m de by double implic tion. First we prove the direct implic tion. We h ve th t the system is schedul ble which me ns there is schedule S which s tisfies ll the l tency constr ints $s_{\mathbf{B}}$ $s_{\mathbf{A}} + C_{\mathbf{B}}$ $L_{\mathbf{AB}}$ $L(A, B)$. But $s_{\mathbf{B}}$ $s_{\mathbf{A}} + C_{\mathbf{B}}$ is the sum of comput tion times of oper tions which re scheduled from $s_{\mathbf{A}}$ to $s_{\mathbf{B}}$. Bec use of the p rti l order the set of the oper tions scheduled from $s_{\mathbf{A}}$ to $s_{\mathbf{B}}$ cont ins t le st some oper tions belonging to $\mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}})$, i $1, 2, \ldots, n$. The oper tions of $\mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}})$ which re scheduled from $s_{\mathbf{A}}$ to $s_{\mathbf{B}}$ re given by scheduling order of oper tions $C_{\mathbf{i}}$ $D_{\mathbf{i}}$ A B i $1, 2, \ldots, n$ in schedule S. Therefore we h ve to study ll the possible orders for these two p irs of oper tions i.e. six orders such th t A is lw ys scheduled before B nd C before D. Also mong the six orders bec use (A, B) is in rel tion X with $(C_{\mathbf{i}}, D_{\mathbf{i}})$ the orders $s_{\mathbf{A}} + C_{\mathbf{A}}$ $s_{\mathbf{B}} + C_{\mathbf{B}}$ $s_{\mathbf{C}_i} + C_{\mathbf{C}_i}$ $s_{\mathbf{D}_i} + C_{\mathbf{D}_i}$ nd $s_{\mathbf{C}_i} + C_{\mathbf{C}_i}$ $s_{\mathbf{D}_i} + C_{\mathbf{D}_i}$ $s_{\mathbf{A}} + C_{\mathbf{A}}$ $s_{\mathbf{B}} + C_{\mathbf{B}}$ re forbidden. It rem ins 4 possible c ses for scheduling the p irs (A, B) nd $(C_{\mathbf{i}}, D_{\mathbf{i}})$, i $1, 2, \ldots, n$

1. $s_{\mathbf{A}} + C_{\mathbf{A}} \quad s_{\mathbf{C}_i} + C_{\mathbf{C}_i} \quad s_{\mathbf{B}} + C_{\mathbf{B}} \quad s_{\mathbf{D}_i} + C_{\mathbf{D}_i},\ i \quad 1, 2, \ldots, j$;
2. $s_{\mathbf{C}_i} + C_{\mathbf{C}_i} \quad s_{\mathbf{A}} + C_{\mathbf{A}} \quad s_{\mathbf{D}_i} + C_{\mathbf{D}_i} \quad s_{\mathbf{B}} + C_{\mathbf{B}},\ i \quad j, \ldots, k$;
3. $s_{\mathbf{C}_i} + C_{\mathbf{C}_i} \quad s_{\mathbf{A}} + C_{\mathbf{A}} \quad s_{\mathbf{B}} + C_{\mathbf{B}} \quad s_{\mathbf{D}_i} + C_{\mathbf{D}_i},\ i \quad k+1, \ldots, m$;
4. $s_{\mathbf{A}} + C_{\mathbf{A}} \quad s_{\mathbf{C}_i} + C_{\mathbf{C}_i} \quad s_{\mathbf{D}_i} + C_{\mathbf{D}_i} \quad s_{\mathbf{B}} + C_{\mathbf{B}},\ i \quad m+1, \ldots, n$;

Bec use of lemm 3 the c se 1 implies th t ll oper tions belonging to $\mathcal{M}(A, E), E \quad \Gamma^-_{\mathbf{C}_i;\mathbf{D}_i}(AB)$, $i \quad 1, 2, \ldots, j$ re scheduled from A to C. Consequently bec use $C_{\mathbf{i}}$ is scheduled before B the rest of oper tions belonging to $\mathcal{M}(A, B)$ i.e. oper tions belonging to $\mathcal{M}(E, B), E \quad ft_{\mathbf{C}_i\mathbf{D}_i}(A, B)$ re scheduled between $C_{\mathbf{i}}$ nd $D_{\mathbf{i}}$, $i \quad 1, 2, \ldots, j$.

Bec use of lemm 4 the c se 1 implies th t ll oper tions belonging to $\mathcal{M}(E, D_{\mathbf{i}}), E \quad \Gamma^+_{\mathbf{AB}}(C_{\mathbf{i}}, D_{\mathbf{i}})$, $i \quad 1, 2, \ldots, j$ re scheduled from B to $D_{\mathbf{i}}$. Consequently bec use $C_{\mathbf{i}}$ is scheduled before B the rest of oper tions belonging to $\mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}})$ i.e. oper tions belonging to $\mathcal{M}(C_{\mathbf{i}}, E), E \quad lt_{\mathbf{AB}}(C_{\mathbf{i}}, D_{\mathbf{i}})$ re scheduled between A nd B.

Bec use of lemm 3 the c se 2 implies th t ll oper tions belonging to $\mathcal{M}(C_{\mathbf{i}}, E), E \quad \Gamma^-_{\mathbf{AB}}(C_{\mathbf{i}}, D_{\mathbf{i}})$, $i \quad j+1, \ldots, k$ re scheduled from $C_{\mathbf{i}}$ to A. Consequently bec use A is scheduled before $D_{\mathbf{i}}$ the rest of oper tions belonging to $\mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}})$ i.e. oper tions belonging to $\mathcal{M}(E, D_{\mathbf{i}}), E \quad ft_{\mathbf{AB}}(C_{\mathbf{i}}, D_{\mathbf{i}})$ re scheduled between A nd B.

Bec use of lemm 4 the c se 2 implies th t ll oper tions belonging to $\mathcal{M}(E, B), E \quad \Gamma^+_{\mathbf{C}_i\mathbf{D}_i}(A, B)$, $i \quad j+1, \ldots, k$ re scheduled from $C_{\mathbf{i}}$ to $D_{\mathbf{i}}$. Consequently bec use A is scheduled before $D_{\mathbf{i}}$ the rest of oper tions belonging to $\mathcal{M}(A, B)$ i.e. oper tions belonging to $\mathcal{M}(A, E), E \quad lt_{\mathbf{C}_i;\mathbf{D}_i}(A, B)$ re scheduled between C nd D.

The c se 3 implies th t ll oper tions belonging to $\mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}})$, $i \quad k+1, \ldots, m$ re scheduled from A to B. The c se 3 does not llow to s y if oper tions belonging to $\mathcal{M}(A, B)$ re scheduled from $C_{\mathbf{i}}$ to $D_{\mathbf{i}}$.

The c se 4 implies th t ll oper tions belonging to $\mathcal{M}(A, B)$ re scheduled from $C_{\mathbf{i}}$ to $D_{\mathbf{i}}$, $i \quad m+1, \ldots, n$. The c se 4 does not llow to s y if oper tions belonging to $\mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}})$ re scheduled from A to B.

In conclusion the set of oper tions which re scheduled from A to B cont ins t le st the oper tions belonging to

$$= \mathcal{M}(A, B) \bigcup \bigcup_{\mathbf{i} \in \{1;\ldots;\mathbf{j}\}; \mathbf{E} \in \mathbf{lt}_{AB}(\mathbf{C}_i;\mathbf{D}_i)} \mathcal{M}(C_{\mathbf{i}}, E) \bigcup \bigcup_{\mathbf{i} \in \{\mathbf{j}+1;\ldots;\mathbf{k}\}; \mathbf{E} \in \mathbf{ft}_{AB}(\mathbf{C}_i;\mathbf{D}_i)} \mathcal{M}(E, D_{\mathbf{i}}) \bigcup \bigcup_{\mathbf{i} \in \{\mathbf{k}+1;\ldots;\}} \mathcal{M}(C_{\mathbf{i}}, D_{\mathbf{i}})$$ where

$j, k, m \quad \mathbb{N}$ re the n tur l numbers for which the following rel tions re s tisfied

$$s_{\mathbf{D}_i} \quad s_{\mathbf{C}_i} + C_{\mathbf{D}_i} \sum_{\mathbf{H} \in \{\mathcal{M}(\mathbf{C}_i;\mathbf{D}_i) \bigcup \bigcup_{\mathbf{E} \in \mathbf{ft}_{C_i D_i}(\mathbf{A};\mathbf{B})} \mathcal{M}(E, B)} \mathrm{C}_{\mathbf{H}} \quad i \quad 1, \ldots, j$$

$$s_{\mathbf{D}_i} \quad s_{\mathbf{C}_i} + C_{\mathbf{D}_i} \quad \sum_{\mathbf{H} \in \{\mathcal{M}(\mathbf{C}_i;\mathbf{D}_i) \bigcup \bigcup_{\mathbf{E} \in \mathbf{lt}_{C_i D_i}(\mathbf{A};\mathbf{B})\}} \mathcal{M}(A,E)} \mathrm{C}_{\mathbf{H}} \quad i \quad j+1,\ldots,k$$

$$s_{\mathbf{D}_i} \quad s_{\mathbf{C}_i} + C_{\mathbf{D}_i} \quad \sum_{\mathbf{H} \in \mathcal{M}(\mathbf{C}_i;\mathbf{D}_i)} C_{\mathbf{H}} \quad i \quad k+1,\ldots,m \quad \text{nd}$$

$$s_{\mathbf{D}_i} \quad s_{\mathbf{C}_i} + C_{\mathbf{D}_i} \quad \sum_{\mathbf{H} \in \{\mathcal{M}(\mathbf{C}_i;\mathbf{D}_i) \bigcup \bigcup_{\mathbf{E} \in \mathcal{M}(\mathbf{A};\mathbf{B})}} C_{\mathbf{H}} \quad i \quad m+1,\ldots,n \ .$$

Using $L(A,B) \quad s_{\mathbf{D}_i} \quad s_{\mathbf{C}_i} + C_{\mathbf{D}_i}$ we obt in inequ lity (12). The direct implic tion is proved.

We prove the inverse implic tion. We h ve inequ lity (12) s tisfied for e ch p ir (A,B) on which l tency constr int is imposed. This me ns th t we m y schedule from A to B the oper tions belonging to

$$= \mathcal{M}(A,B) \bigcup \bigcup_{\mathbf{i} \in \{1;:::;\mathbf{j}\};\mathbf{E} \in \mathbf{lt}_{AB}(\mathbf{C}_i;\mathbf{D}_i)} \mathcal{M}(C_{\mathbf{i}},E) \bigcup \bigcup_{\mathbf{i} \in \{\mathbf{j}+1;:::;\mathbf{k}\};\mathbf{E} \in \mathbf{f}\,\mathbf{t}_{AB}(\mathbf{C}_i;\mathbf{D}_i)} \mathcal{M}(E,D_{\mathbf{i}}) \bigcup$$

$\bigcup_{\mathbf{i} \in \{\mathbf{k}+1;:::;\}} \mathcal{M}(C_{\mathbf{i}},D_{\mathbf{i}})$. The set cont ins ll the oper tions belonging to $\mathcal{M}(C_{\mathbf{i}},D_{\mathbf{i}}) \quad i$ $1,\ldots,n$ which must be scheduled from A to B in order to s tisfy the p rti l order. Using inequ lity (12) for e ch p ir (A,B) we obt in schedule which s tisfies the p rti l order. Bec use l tency constr ints re lre dy s tisfied by inequ lity (12) the system is schedul ble. The inverse implic tion is proved nd the theorem is proved ⊡

Remark 8 *The theorem 5 presents schedul bility conditions which llows to obt in minim l sets corresponding to the l tency constr ints. Bec use building these sets for gener l c se is prob bly NP-h rd problem, we will consider from now on only c ses where e ch l tency constr int is in rel tion* X *with t most one l tency constr int. For this c se in the next section we give polynomi l scheduling lgorithm.*

We give n ex mple of system with l tency constr ints in rel tion X.

Example 6 *Let* $= (\ ,\mathcal{E})$ *be the gr ph given in gure 15 with the execution times* $C_{\mathbf{A}} =$ $1,\ A$ *nd the l tency constr ints* $L_{\mathbf{AC}} = 3$, $L_{\mathbf{DF}} = 9$ *nd* $L_{\mathbf{HJ}} = 5$. *We h ve th t* (A,C) X (D,F) *nd* (D,F) X (H,J). *The schedul bility conditions re s tis ed nd the gure 16 gives schedule which s tis es ll constr ints.*

4.4 General schedulability condition

Giving gener l schedul bility condition mounts to prove th t there is schedule which s tisfies ll the l tency constr ints if nd only if e ch l tency constr int in rel tion II Z or X with other l tency constr ints is s tisfied. Previously we obt ined sep r tely schedul bility condition for systems with ll p irs in rel tion II nd Z nd schedul bility condition for

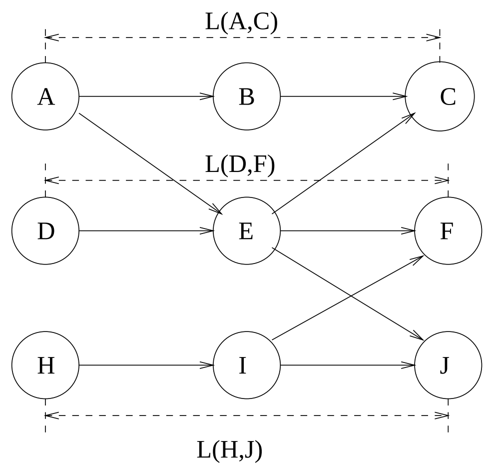


Figure 15 System with l tency constr ints in rel tion X

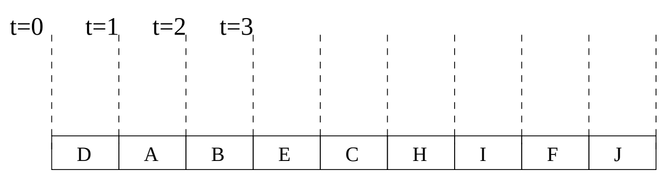


Figure 16 Schedule s tisfying ll constr ints

systems with ll p irs in rel tion X. From now on we will study only the schedul bility condition of systems with ll l tency constr int $L_{\mathsf{X;Y}}$ gre ter th n the sum of comput tion times of oper tions belonging to $\mathcal{M}(X,Y)$ bec use the systems which do not s tisfy this l tter condition re obviously not schedul ble. The following theorem gives the gener l schedul bility condition.

Bec use ll l tency constr int $L_{\mathsf{X;Y}}$ re gre ter th n the sum of comput tion times of oper tions belonging to $\mathcal{M}(X,Y)$. Bec use of lemm 1 we h ve th t schedule in which between s_{A} nd s_{B} there re no oper tion belonging to p irs in rel tion II or Z with (A,B) does not s tisfy the l tency constr ints with ll p irs in rel tion II nd X on which l tency constr ints re imposed we proved th t system with ll the l tency constr ints well-defined is lw ys schedul ble. This rem ining true for the gener l c se the gener l schedul bility condition will concerns only p irs of oper tions which re in rel tion.

The rem 6 *Let $= (,\mathcal{E})$ be gr ph de ning system nd $\mathcal{L}$ set of l tency constr ints imposed on p irs of oper tions. There is schedule s tisfying ll the l tency constr ints if nd only if inequ lity (12) is s tis ed for e ch p ir (A,B) in rel tion* X *with n p irs $(C_{\mathsf{i}}, D_{\mathsf{i}})$, i $1,2,\ldots n$.*

Pr f The proof is m de by double implic tion nd follows the s me principles s the proof of theorem 5. The direct implic tion ssumes th t the system is schedul ble nd proves th t inequ lity (12) is s tisfied t king into ccount for e ch p ir on which l tency constr int is imposed ll p irs such th t this p ir is in rel tion II Z or X. The inverse implic tion builds schedule which s tisfies ll l tency constr ints using inequ lity (12) ⊡

Remark 9 *For system with n l tency constr ints, the number of inequ lities of type(12) which must be veri ed is t most $n(n \quad 1)$.*

Example 7 *For the gr ph of gure 17, we h ve ll comput tion times equ l to 1, $C_{\mathsf{A}} = 1$, A . The following l tency constr ints re imposed: $L_{\mathsf{AB}} = 7$, $L_{\mathsf{CD}} = 12$, $L_{\mathsf{MF}} = 3$ nd $L_{\mathsf{OR}} = 4$. Note th t $(A,B)\mathsf{X}(C,D)$, $(A,B)\mathsf{Z}(M,F)$, $(C,D)\mathsf{X}(M,F)$ nd (O,R) is in rel tion* II *with ll the other l tency constr ints. First, note th t ll l tency constr ints $L_{\mathsf{X;Y}}$ re gre ter th n the sum of comput tion times of oper tions belonging to $\mathcal{M}(X,Y)$. Then pplying theorem 6, the system is schedul ble if nd only if the following rel tions re s tis ed:*

$$L_{\mathsf{AB}} \qquad \sum_{\mathsf{X} \in \{\mathcal{M}(\mathsf{A;B}) \cup \bigcup_{\mathsf{Y} \in \mathsf{lt}_{AB}(\mathsf{C;D})} \mathcal{M}(C,Y)} C_{\mathsf{X}}$$

$$L_{\mathsf{CD}} \qquad \sum_{\mathsf{X} \in \{\mathcal{M}(\mathsf{C;D}) \bigcup \bigcup_{\mathsf{Y} \in \mathsf{f\,t}_{CD}(\mathsf{A;B})} \mathcal{M}(Y,B) \bigcup \mathcal{M}(M,F)} C_{\mathsf{X}}$$

$$L_{\mathsf{MF}} \qquad \sum_{\mathsf{X} \in \mathcal{M}(\mathsf{M;F})} C_{\mathsf{X}}$$

We obt in:

$$\begin{cases} L_{\mathsf{AB}} \quad C_{\mathsf{A}} + C_{\mathsf{G}} + C_{\mathsf{H}} + C_{\mathsf{I}} + C_{\mathsf{B}} + C_{\mathsf{C}} + C_{\mathsf{K}} \\ L_{\mathsf{CD}} \quad C_{\mathsf{C}} + C_{\mathsf{I}} + C_{\mathsf{J}} + C_{\mathsf{K}} + C_{\mathsf{L}} + C_{\mathsf{D}} + C_{\mathsf{M}} + \\ C_{\mathsf{N}} + C_{\mathsf{F}} + C_{\mathsf{H}} + C_{\mathsf{B}} \\ L_{\mathsf{MF}} \quad C_{\mathsf{M}} + C_{\mathsf{N}} + C_{\mathsf{F}} \end{cases}$$

Bec use ll three inequ tions re veri ed, the system is schedul ble. Indeed the schedule $S = \; s_{\mathsf{O}} \;\; s_{\mathsf{P}} \;\; s_{\mathsf{R}} \;\; s_{\mathsf{E}} \;\; s_{\mathsf{A}} \;\; s_{\mathsf{G}} \;\; s_{\mathsf{C}} \;\; s_{\mathsf{K}} \;\; s_{\mathsf{I}} \;\; s_{\mathsf{H}} \;\; s_{\mathsf{B}} \;\; s_{\mathsf{M}} \;\; s_{\mathsf{N}} \;\; s_{\mathsf{F}} \;\; s_{\mathsf{J}} \;\; s_{\mathsf{L}} \;\; s_{\mathsf{D}}$ *s tis es ll the l tency constr ints. In the following section we give n lgorithm which llows to obt in schedule which s tis es ll the l tency constr ints for the c se where e ch l tency constr int is in rel tion* X *with t most one l tency constr int.*

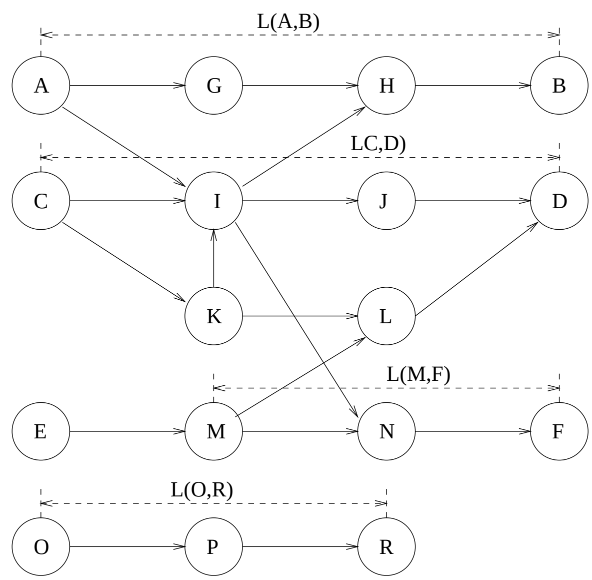


Figure 17 Ex mple for gener l schedul bility condition

5 Optimal scheduling algorithm

This section presents n optim l scheduling lgorithm for the c se where e ch l tency constr int is in rel tion X with t most one l tency constr int. This lgorithm builds schedules s tisfying the schedul bility condition obt ined in the previous section.

A scheduling lgorithm tr nsforms the p rti l order defined by the gr ph in tot l order (one of possibles tot l orders) which s tisfies the l tency constr ints. In our c se the scheduling lgorithm 2 is pplied only to the p ttern nd it uses lgorithm 1 th t we c ll "m rking oper tions lgorithm". This lgorithm " nticip tes" which oper tions re the most import nt for the l tency constr ints.

We denote by $= (\ , \mathcal{E})$ the oriented cyclic gr ph-p ttern where is the set of oper tions nd $\mathcal{E}$ is the set of edges nd by $\mathcal{D}(A) =$ B such th t t le st p th from A to B . We denote by the working-set used during the lgorithm.

The purpose of the m rking lgorithm is to ssoci te to e ch oper tion A m rk which indic tes if n oper tion C belonging to p ir (B, C) with l tency constr int will be scheduled fter A. This is possible th nks to the existence of t le st one p th between A nd C nd to the non-existence of ny p th between A nd B. If for n oper tion there re sever l oper tions s tisfying this property then the m rk is equ l to the sm llest l tency constr int. We denote this m rk by $mark(A)$ $\mathbb{N}^* \bigcup$ where is positive integer gre ter th n ny other number belonging to $\mathbb{N}^*$. These m rks m y ch nge their v lues during the m rking lgorithm. This l tter is pplied only on the p ttern of the gr ph like the scheduling lgorithm.

Alg rithm 1

Initialisati n if (A, B) h s l tency constr int $L(A, B)$ then $mark(B) = L(A, B)$ nd $mark(A) =$ else $mark(A) = mark(B) =$. Moreover if B belongs to sever l p irs with l tency constr ints then $mark(B) = min_{(\mathbf{C};\mathbf{B})\in\mathcal{L}}\ L(C, B)$ nd $mark(A) =$. Let $= \mathcal{L}$.

Step 1 for (A, B) nd for e ch oper tion C V A, B there re three possibilities

( ) if A $\mathcal{D}(C)$ then $mark(C) = mark(C)$;
(b) if $A \ / \ \mathcal{D}(C)$ nd B $\mathcal{D}(C)$ then $mark(C) = min(mark(C), mark(B))$;
(c) if $A \ / \ \mathcal{D}(C)$ nd $B \ / \ \mathcal{D}(C)$ then $mark(C) = mark(C)$.

Let $=$ (A, B) .

Step 2 if $=$ then goto Step 1 else the lgorithm stops.

We denote by $\mathcal{L}_{\mathsf{X}}$ the set of l tency constr ints which re in rel tion X with other l tency constr ints nd which do not h ve the di erence between the v lue of the l tency nd the sum of oper tions belonging to the p ir is equ l to 0. Without ny loss of gener lity we consider this set ordered i.e. if $L(A, B)$ is before $L(C, D)$ in $\mathcal{L}_{\mathsf{X}}$ then $L_{\mathsf{AB}} < L_{\mathsf{CD}}$.

The scheduling lgorithm uses the m rks obt ined using the m rking lgorithm. We denote by the working set by s_{T} the st rt time of the l st scheduled oper tion by C_{T} its execution time nd by $Prec(A)$ the set of predecessors of oper tion A. During the pplic tion of the lgorithm the set cont ins the v il ble oper tions. We notice th t between two oper tions which re v il ble in the s me time there is no p th. Also during the pplic tion of the lgorithm e ch time th t n oper tion is scheduled s_{T} (respectively C_{T}) is repl ced by the st rt time (execution time) of this oper tion.

Alg rithm 2

Initialisati n $= \bigcup_{\mathbf{A} \in \mathbf{V} \ \ and \ \mathbf{P\,rec}(\mathbf{A})=\emptyset} A$ nd $s_{\mathbf{T}} = 0, C_{\mathbf{T}} = 0$.

Step 1 *(oper tion with l tency constr int)*

if A such th t $mark(A) =$ when we chose A such th t

$$mark(A) = min_{\mathbf{B} \in \mathcal{W}} \ mark(B)$$

If $L(C_1, D_1)$ $\mathcal{L}_{\mathbf{X}}$ nd $L(C_{\mathbf{i}}, D_{\mathbf{i}})$ $\mathcal{L}_{\mathbf{X}}$, $i = 2, 3, \ldots n$ such th t A $\bigcup_{\mathbf{E} \in \Gamma^{+}_{C_i D_i}(\mathbf{C}_1;\mathbf{D}_1)} \mathcal{M}(E, D_1)$, $i = 2, 3, \ldots n$ then let B be the oper tion with the sm llest m rk such th t i $2, 3, \ldots, n$ with B $\bigcup_{\mathbf{E} \in \Gamma^{+}_{C_1 D_1}(\mathbf{C}_i;\mathbf{D}_i)} \mathcal{M}(E, D_{\mathbf{i}})$. If the oper tion B exists then $s_{\mathbf{B}} = s_{\mathbf{T}} + C_{\mathbf{T}}$ we remove B from ll oper tions which become v il ble re dded to nd we go to step 4. If the oper tion B does not exist then $s_{\mathbf{A}} = s_{\mathbf{T}} + C_{\mathbf{T}}$ we remove A from ll oper tions which become v il ble re dded to nd we go to Step 4.

Step 2 *(oper tion which is not rst in l tency constr int)*

if A such th t there is no oper tion B with (A, B) $\mathcal{L}$ then $s_{\mathbf{A}} = s_{\mathbf{T}} + C_{\mathbf{T}}$ we remove it from ll oper tions which become v il ble re dded to nd we go to Step 4.

Step 3 *(oper tion which is rst in l tency constr int)*

if A such th t there is B V vec (A, B) $\mathcal{L}$ lors $s_{\mathbf{A}} = s_{\mathbf{T}} + C_{\mathbf{T}}$ with $L(A, B) = max_{(\mathbf{C};\mathbf{D}) \in \mathcal{L} \ \ et \ \mathbf{C} \in \mathcal{W}} \ L(C, D)$ we remove it from ll oper tions which become v il ble re dded to nd we go to Step 4.

Step 4

if $=$ then the lgorithm stops else we go to Step 1.

The scheduling lgorithm 2 is proved optim l for the c se where e ch l tency constr int is in rel tion X with t most one l tency constr int. Before proving this optim lity we give n ex mple of schedule obt ined with this lgorithm.

Example 8 *Let* $= (\ , \mathcal{E})$ *be the gr ph given in gure 1 with* $C_{\mathbf{A}} = 2$, $C_{\mathbf{B}} = 1$ *nd* $C_{\mathbf{C}} = 2$ *nd the l tency constr ints* $L(A_2, C_2) = 10$ *nd* $L(B, C_1) = 9$. *After pplying the m rking lgorithm we obt in the m rks given in t ble 1 where* $m(A)$ *denotes* $mark(A)$.

We obt in the t ble 2 using the scheduling lgorithm 2.

The rem 7 *The scheduling lgorithm 2 is optim l (if there is schedule which s tis es the constr ints then the lgorithm will nd it).*

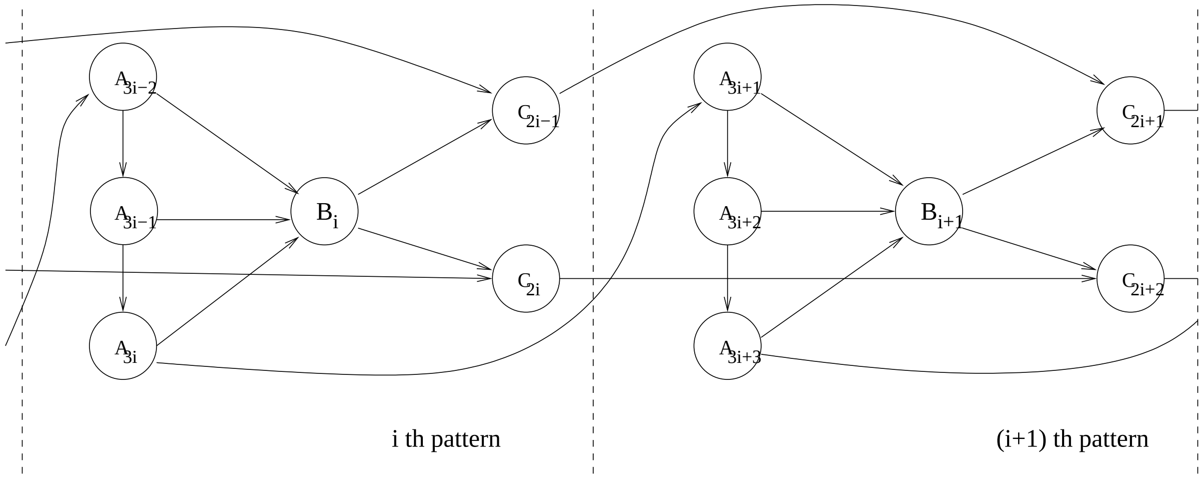


Figure 18 Gr ph illustr ting the scheduling lgorithm 2

	$m(A_1)$	$m(A_2)$	$m(A_3)$	$m(B)$	$m(C_1)$	$m(C_2)$
Initi lis tion					9	10
$L(B, C_1)$					9	10
$L(A_2, C_2)$			10	10	9	10

T ble 1 M rks obt ined using the m rking lgorithm

	$s_{\mathbf{T}}$	$C_{\mathbf{T}}$	Step used utilisé	Choice
A_1	0	0	Step 2	$s_{\mathbf{A}_1} = 0$
A_2	0	2	Step 3	$sA_2 = 2$
A_3	2	2	Step 1	$s_{\mathbf{A}_3} = 4$
B	4	2	Step 1	$s_{\mathbf{B}} = 6$
C_1, C_2	6	1	Step 1	$s_{\mathbf{C}_2} = 7$
C_1	7	2	Step 1	$s_{\mathbf{C}_1} = 9$

T ble 2 Schedule obt ined using the lgorithm 2

Pr f We prove th t the scheduling lgorithm builds 2 schedules which s tisfy the schedu- l bility conditions nd if these schedules do not s tisfy the constr ints then no other will. The proof h s three p rts first one concerning the l tency constr ints imposed on p irs in rel tion II second p rt concerning the l tency constr ints imposed on p irs in rel tion Z nd l st p rt concerning the l tency constr ints imposed on p irs in rel tion X.

Without loss of gener lity we consider th t ll l tency constr ints h ve di erent v lues.

Let $L(A, B)$ nd $L(C, D)$ be some two l tency constr ints imposed on p irs in rel tion II. Bec use there is no p th from n oper tion belonging to (A, B) to n oper tion belonging to (C, D) then the m rk of oper tions belonging to (A, B) is not modified by the l tency constr int $L(C, D)$. Likewise bec use there is no p th from n oper tion belonging to (C, D) to n oper tion belonging to (A, B) then the m rk of oper tions belonging to (C, D) is not modified by the l tency constr int $L(A, B)$. Therefore the oper tions belonging to the two p irs h ve di erent m rks nd they will be scheduled either in the following order $s_{\mathsf{A}} + C_{\mathsf{A}} < s_{\mathsf{B}} + C_{\mathsf{B}} < s_{\mathsf{C}} + C_{\mathsf{C}} < s_{\mathsf{D}} + C_{\mathsf{D}}$ either in the order $s_{\mathsf{C}} + C_{\mathsf{C}} < s_{\mathsf{D}} + C_{\mathsf{D}} < s_{\mathsf{A}} + C_{\mathsf{A}} < s_{\mathsf{B}} + C_{\mathsf{B}}$. The two orders le d to the schedul bility condition for the c se of l tency constr ints imposed on p irs in rel tion II.

Let $L(A, B)$ nd $L(C, D)$ be some two l tency constr ints imposed on p irs which re in rel tion Z. Bec use there is no p th from n oper tion belonging to (C, D) to n oper tion belonging to (A, B) then the m rk of oper tions belonging to (C, D) is not modified by the l tency constr int $L(A, B)$. Bec use there is p th from n oper tion belonging to (A, B) to t le st n oper tion belonging to (C, D) then the m rk of A m y be modified by the l tency constr int $L(C, D)$ only if $L_{\mathsf{AB}} > L_{\mathsf{CD}}$. In this c se the oper tions belonging to the two p irs will be scheduled in the following order $s_{\mathsf{A}} + C_{\mathsf{A}} < s_{\mathsf{B}} + C_{\mathsf{B}} < s_{\mathsf{C}} + C_{\mathsf{C}} < s_{\mathsf{D}} + C_{\mathsf{D}}$. In the opposite c se the m rk of A is not modified by the l tency constr int $L(C, D)$ nd the oper tions belonging to the two p irs will be scheduled either in this order $s_{\mathsf{A}} + C_{\mathsf{A}} < s_{\mathsf{B}} + C_{\mathsf{B}} < s_{\mathsf{C}} + C_{\mathsf{C}} < s_{\mathsf{D}} + C_{\mathsf{D}}$ either in this order $s_{\mathsf{C}} + C_{\mathsf{C}} < s_{\mathsf{D}} + C_{\mathsf{D}} < s_{\mathsf{A}} + C_{\mathsf{A}} < s_{\mathsf{B}} + C_{\mathsf{B}}$. The two orders le d to the schedul bility condition for the c se of l tency constr ints imposed on p irs in rel tion Z.

Let $L(A, B)$ nd $L(C, D)$ be some two l tency constr ints imposed on p irs which re in rel tion X. Bec use there is p th from n oper tion belonging to (A, B) to t le st n oper tion belonging to (C, D) then the m rk of A m y be modified by the l tency constr int $L(C, D)$ only if $L_{\mathsf{AB}} > L_{\mathsf{CD}}$. In this c se the existence of p th between oper tions belonging to (C, D) to t le st n oper tion belonging to (A, B) does not this m rk. Likewise we obt in the s me conclusion if $L_{\mathsf{AB}} < L_{\mathsf{CD}}$.

For the three c ses the schedul bility conditions re s tisfied by the schedules obt ined using the lgorithm 2. The theorem is proved ⊡

6 Conclusion and future work

We give schedul bility condition for systems with l tency nd precedence constr ints using the model b sed on gr phs presented in [5]. This result is obt ined by studying three rel tions II Z nd X between p irs of oper tions nd by studying the l tency constr ints imposed on the p irs of oper tions s tisfying the l tter rel tions.

We give scheduling lgorithm which is proved optim l for the c se where the l tency constr ints re in X with t most one other l tency constr int.

Presently we study the complexity of the scheduling problem for the c se where the l tency constr ints m y be in X with t le st one other l tency constr int. We pl n to gener lise this schedul bility condition to re l-time systems with precedence periodicity nd l tency constr ints.

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